

Explicit tough Ramsey graphs

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Dedicated to our parents on the occasion of their 30th anniversary.

Abstract

A graph G is t -tough if any induced subgraph of it with $x > 1$ connected components is obtained from G by deleting at least tx vertices. Chvátal conjectured that there exists an absolute constant t_0 so that every t_0 -tough graph is pancyclic. This conjecture was disproved by Bauer, van den Heuvel and Schmeichel by constructing a t_0 -tough triangle-free graph for every real t_0 . For each finite field \mathbb{F}_q with q odd, we consider graphs associated to the finite Euclidean plane and the finite upper half plane over \mathbb{F}_q . These graphs have received serious attention as they have been shown to be Ramanujan (or asymptotically Ramanujan) for large q . We will show that for infinitely many q , these graphs provide further counterexamples to Chvátal's conjecture. They also provide a good constructive lower bound for the Ramsey number $R(3, k)$.

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1. Introduction

The *toughness* $t(G)$ of a graph G is the largest real t so that for every positive integer $x \geq 2$ one must delete at least tx vertices from G in order to get an induced subgraph of it with at least x connected components. G is t -tough if $t(G) \geq t$. This parameter was introduced by Chvátal in [10]. Chvátal proposed the following conjecture.

Conjecture 1 ([10]) *A simple unlabeled graph on n vertices is called pancyclic if it contains cycles of all lengths $3, 4, \dots, n$. Then there exists an absolute constant t_0 such that every t_0 -tough graph is pancyclic.*

This conjecture was disproved by Bauer, van den Heuvel and Schmeichel [7] who constructed, for every real t_0 , a t_0 -tough triangle-free graph. They defined a sequence of triangle-free graphs H_1, H_2, H_3, \dots with $|V(H_j)| = 2^{2j-1}(j+1)!$ and $t(H_j) \geq \sqrt{2j+4}/2$. We present here two new explicit constructions based on finite analogues of Euclidean and non-Euclidean planes.

The first construction is based on a remarkable new approach of Wildberger to trigonometry and Euclidean geometry. In [29], Wildberger replaces distance by quadrance and angle by spread, thus allowing the development of Euclidean geometry over any field. The following definition follows from [29].

Definition 1 *Let \mathbb{F}_q be a finite field of order q where q is an odd prime power. The quadrance $Q(X, Y)$ between the points $X = (x_1, x_2)$, and $Y = (y_1, y_2)$ in \mathbb{F}_q^2 is the number*

$$Q(X, Y) = (y_1 - x_1)^2 + (y_2 - x_2)^2. \quad (1)$$

Let q be an odd prime power and \mathbb{F}_q be the finite field with q elements. We can associate graphs to the symmetric plane \mathbb{F}_q^2 using the quadrance.

Definition 2 For a fixed $a \in \mathbb{F}_q$, the quadrance graph $D_q(a)$ has the vertex set \mathbb{F}_q^2 , and $X, Y \in \mathbb{F}_q^2$ are adjacent if and only if $Q(X, Y) = a$.

This graph or the so-called finite Euclidean graph was first studied by Moorhouse [18] and then by Medrano et al [17]. Note that for any $a, b \neq 0 \in \mathbb{F}_q$ then $D_q(a)$ and $D_q(b)$ are isomorphic. In usual 2-dimensional Euclidean space \mathbb{R}^2 , then the quadrance between X, Y is unity if and only if the distance between X, Y is unity, so the unit-quadrance graph $D_q(1)$ becomes the unit-distance graph.

The second construction is based on the well-known finite upper half planes constructed in a similar way using an analogue of Poincaré's non-Euclidean distance. We follow the construction in [22]. Let \mathbb{F}_q be the finite field with $q = p^r$ elements, where p is an odd prime. Suppose σ is a generator of the multiplicative group \mathbb{F}_q^* of nonzero elements in \mathbb{F}_q . The extension $\mathbb{F}_q(\sigma)$ is analogous to $\mathbb{C} = \mathbb{R}[i]$. We define the *finite Poincaré upper half-plane* as

$$H_q = \{z = x + y\sqrt{\sigma} : x, y \in \mathbb{F}_q \text{ and } y \neq 0\}. \quad (2)$$

Note that “half-plane” is something of a misnomer since $y \neq 0$ may not be a good finite analogue of the condition $y > 0$ that defines the usual Poincaré upper half-plane in \mathbb{C} . In fact, H_q is more like a double covering of a finite upper half-plane. We use the familiar notation from complex analysis for $z = x + y\sqrt{\sigma} \in H_q$: $x = \text{Re}(z)$, $y = \text{Im}(z)$, $\bar{z} = x - y\sqrt{\sigma} = z^q$, $N(z) = \text{Norm of } z = z\bar{z} = z^{1+q}$. The *Poincaré distance* between $z, w \in H_q$ is

$$d(z, w) = \frac{N(z - w)}{\text{Im}(z)\text{Im}(w)}. \quad (3)$$

This distance is not a metric in the sense of analysis, but it is $GL(2, \mathbb{F}_q)$ -invariant: $d(gz, gw) = d(z, w)$ for all $g \in GL(2, \mathbb{F}_q)$ and all $z, w \in H_q$. We can attach graphs to H_q by a method analogous to that which led to the unit-quadrance graphs D_q .

Definition 3 For a fixed $a \in \mathbb{F}_q$, the finite non-Euclidean graph $V_q(\sigma, a)$ has vertices as the points in H_q and edges between vertices z, w if and only if $d(z, w) = a$.

Except when $a = 0$ or $a = 4\sigma$, $P_q(\sigma, a)$ is a connected $(q + 1)$ -regular graph. When $a = 0, 4\sigma$ then $P_q(\sigma, a)$ is disconnected, with one or two nodes, respectively, per connected component. As a varies, we have $q - 2$ $(q + 1)$ -regular graphs $P_q(\sigma, a)$. The question of whether these graphs are always nonisomorphic or not is still open. These graphs are studied extensively in the literature, see [1-2, 8, 9, 11, 12, 20-22].

The rest of this paper is organized as follows. In Section 2 we determine the girth, diameter and triangle-freeness of these graphs. In Section 3 we establish some useful facts about toughness and cutsets of graphs that we will need in the study of quadrance graphs and finite non-Euclidean graphs. We then show in Sections 4 and 5 that for infinitely many values of q , these graphs constitute counterexamples to Chvátal's conjecture and also provide a good constructive lower bound for the Ramsey number $R(3, k)$. Finally, we make some further remarks on the chromatic numbers of these graphs.

2. Girth and diameter

We define the girth to be the length of a shortest circuit in a graph. The diameter is the maximum length of shortest paths between two vertices of a graph. The main aim of this section is to determine the girth and diameter of finite Euclidean and non-Euclidean graphs.

We first consider the finite Euclidean graph D_q .

Definition 4 The circle $C_k(A_0)$ in \mathbb{F}_q^2 with center $A_0 \in \mathbb{F}_q \times \mathbb{F}_q$ and quadrance k is the set of all points $X \in \mathbb{F}_q \times \mathbb{F}_q$ such that

$$Q(A_0, X) = k.$$

The following lemma ([27, 29]) gives us the number of intersections between any two circles in \mathbb{F}_q^2 .

Lemma 1 Let $i, j \neq 0$ in \mathbb{F}_q and let X, Y be two distinct points in \mathbb{F}_q^2 such that $Q(X, Y) = k \neq 0$. Then $|C_i(X) \cap C_j(Y)|$ only depends on i, j and k . Precisely, let

$$f(i, j, k) = ij - (k - i - j)^2/4.$$

Then the number of intersection points is p_{ij}^k , where

$$p_{ij}^k = \begin{cases} 0 & \text{if } f(i, j, k) \text{ is a non-square,} \\ 1 & \text{if } f(i, j, k) = 0, \\ 2 & \text{if } f(i, j, k) \text{ is a square.} \end{cases} \quad (4)$$

If -1 is a square in \mathbb{F}_q then there exist distinct points X, Y in \mathbb{F}_q^2 with $Q(X, Y) = 0$. The following lemma gives us the intersection number of two circles $C_i(X)$ and $C_j(Y)$ in this case.

Lemma 2 ([27]) *For any $i, j \neq 0$ in \mathbb{F}_q . Suppose that X, Y are two distinct points in \mathbb{F}_q^2 with $Q(X, Y) = 0$. Then the circle $C_i(X)$ intersects $C_j(Y)$ if and only if $i \neq j$. Furthermore, if $i \neq j$ then two circles intersect at only one point.*

The proof of Lemma 2 is similar to the proof of Lemma 1.

Theorem 1 *For a fixed $a \neq 0 \in \mathbb{F}_q$, the girth of quadrance graph $Q_q(a)$ is 3 if 3 is a square in \mathbb{F}_q and 4 otherwise.*

Proof We have $\{(0, 0), (0, 1), (1, 1), (1, 0)\}$ is a cycle of length 4 in $Q_q(1)$ so the girth of unit-quadrance graph is at most 4. Besides, $Q_q(a), Q_q(b)$ are isomorphic for any $a, b \neq 0 \in \mathbb{F}_q$, so all quadrance graphs $Q_q(a)$ have girth at most 4. From Lemma 1, the girth of quadrance graph $Q_q(a)$ is 3 if and only if $f(a, a, a) = 3a^2/4$ is a square in \mathbb{F}_q or 3 is a square in \mathbb{F}_q . This concludes the proof of the theorem. \square

Theorem 2 *If $q \equiv 3 \pmod{4}$ then D_q has diameter 3. Otherwise, $D_q(a)$ has diameter 3 or 4.*

Proof Since all quadrance graphs are isomorphic, we only need to prove the theorem for unit-quadrance graph $D_q = D_q(1)$. Define $g(x) = 4f(x, 1, 1) = (4 - x)x$. Then there exist $u, v \neq 0 \in \mathbb{F}_q$ such that $g(u)$ is a square in \mathbb{F}_q while $g(v)$ is not. Choose any $X \in C_v((0, 0))$. By Lemma 1, X is not reachable from $(0, 0)$ by two unit-steps. Thus, the diameter of D_q is at least 3.

Now suppose that $q \equiv 3 \pmod{4}$. Then D_q is a $(q + 1)$ -regular graph. For any vertex $X \in D_q$, we define the set $V(X)$ to be all vertices that are adjacent to X and $T(X)$ to be all the vertices that are connected by a path of at most two edges from X . Then

$$T(X) = \bigcup_{Y \in V(X)} V(Y). \quad (5)$$

By Lemma 1, a fixed vertex W can be in $V(Y)$ for at most 2 vertices $Y \in V(X)$. Thus, $|T(X)| \geq (q + 1)q/2$. This implies that $T(X)$ contains elements from no fewer than $\lceil q/2 \rceil$ circles $C_i(X)$. Thus, there are no more than $q - (q + 1)/2 = (q - 1)/2$ circles outside of $T(X)$. Let Z be any point then Z is adjacent to at most $q - 1$ points that are not in $T(X)$. Thus Z is connected to X by a path of length at most three. This implies that D_q has diameter 3.

Suppose that $q \equiv 1 \pmod{4}$. For any two points X, Y with $Q(X, Y) \neq 1$, by Lemma 2 we can choose a point Z such that $Q(X, Z) = 1, Q(Y, Z) = 0$. By Lemma 2 again, we can choose a point W such that $Q(Z, W) = 1$ and $Q(W, Y) = u$. Since $g(u)$ is a square in \mathbb{F}_q , by Lemma 1, we can choose a point T such that $Q(W, T) = Q(T, Y) = 1$. Thus, Y is connected to X by a path of length at most four. If $Q(X, Y) = 1$ then Y is adjacent to X in the graph. This concludes the proof of the theorem. \square

The girth and diameter of finite non-Euclidean graphs have been studied before. The following theorem is due to Celniker [8].

Theorem 3 ([8]) *Let q be odd and $a \notin \{0, 4\sigma\}$. The girth of $V_q(\sigma, a)$ is either 3 or 4. Furthermore, the girth is 3 if $a = 2\sigma$ and $q \equiv 3 \pmod{4}$ or if a and $a - 3\sigma$ are squares in \mathbb{F}_q . The girth is 4 if $a = 2\sigma$ and $q \equiv 1 \pmod{4}$.*

In [2], Angel and Evans obtained the following result for the diameter of finite non-Euclidean graphs $V_q(\sigma, a)$.

Theorem 4 ([2]) *Let q be odd and $a \notin \{0, 2\sigma, 4\sigma\}$. Then $V_q(\sigma, a)$ has diameter 3 or 4 according to whether $\sigma - a$ is a square or a non-square in \mathbb{F}_q . Besides, the diameter of $X_{2\sigma}$ is 3, unless $q = 3$ or 5, in which case the diameter is 2.*

3. Toughness and cutsets

Let G be a d -regular graph with n eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. It is well-known that the largest eigenvalue $\lambda_1 = d$. We know that for $a \neq 0$ the quadrance graph $D_q(a)$ is a regular graph with degree $\Delta(D_q(a)) = q - (-1)^{(q-1)/2}$ (see [17]). Medrano et al. [17] give a general bound for eigenvalue of $D_q(a)$.

Lemma 3 [17] *Let $\lambda \neq \Delta(D_q(a))$ be any eigenvalue of graph $D_q(a)$. Then*

$$|\lambda| \leq 2q^{1/2}. \quad (6)$$

The finite non-Euclidean graphs $V_q(\sigma, a)$ are also known as $(q+1)$ -regular graphs for $a \neq 0, 4\sigma$. By combining works of Weil [28], Evans [11, 12], Katz [14, 15], Li [16] and many others, it was proved that the finite non-Euclidean graphs are all Ramanujan graphs, that is, we have the following bound for eigenvalue of $V_q(\sigma, a)$.

Lemma 4 [22] *Let $\lambda \neq q+1$ be any eigenvalue of graph $V_q(\sigma, a)$. Then*

$$|\lambda| \leq 2q^{1/2}. \quad (7)$$

An (n, d, μ) -graph is a d -regular graph on n vertices, in which every eigenvalue $\mu < d$ satisfies $|\mu| \leq \lambda$. The following result is due to Alon in [4].

Theorem 5 [4] *Let $G = (V, E)$ be an (n, d, λ) -graph. Then the toughness $t = t(G)$ of G satisfies*

$$t > \frac{1}{3} \left(\frac{d^2}{\lambda d + \lambda^2} - 1 \right). \quad (8)$$

Let σ be an arbitrary orientation of graph G , and let D be the incidence matrix of G^σ . Then the Laplacian of G is the matrix $Q(G) = DD^T$. It is easy to show that the Laplacian does not depend on the orientation σ , and hence is well-defined. Let G be a d -regular graph. If the adjacency matrix A of G has eigenvalues $\lambda_1 \geq \dots \geq \lambda_n$, then the Laplacian Q has eigenvalues $\theta_1 = d - \lambda_1 \leq \dots \leq \theta_n = d - \lambda_n$. It is well-known (see [13], pages 287-288) that, if G is a graph on n vertices then the minimum value of

$$\frac{\sum_{uv \in E(G)} (x_u - x_v)^2}{\sum_u x_u^2}, \quad (9)$$

as x ranges over all nonzero vectors orthogonal to $(1, \dots, 1)$, is $\theta_2(G)$. The maximum value is $\theta_n(G)$.

If $S \subset V(G)$, let δS denote the set of edges with one end in S and the other in $V(X) - S$. Suppose that $|S| = s$. Let z be the vector whose value is $n - s$ on the vertices in S and $-s$ on the vertices not in S . Then z is orthogonal to $(1, \dots, 1)$, so we have

$$\theta_2(G) \leq \frac{\sum_{uv \in E(G)} (x_u - x_v)^2}{\sum_u x_u^2} = \frac{|\delta S|n^2}{s(n-s)^2 + (n-s)s^2} = \frac{n|\delta S|}{|S|(n-|S|)} \leq \theta_n(G). \quad (10)$$

The *bisection width* of graph G on n vertices is the minimum value of $|\delta S|$, for any subset S of size $\lfloor n/2 \rfloor$. Let $bip(G)$ denote the maximum number of edges in a spanning bipartite subgraph of G . This equals the maximum value of $|\delta S|$, where S ranges over all subsets of $V(X)$ with size at most $|V(X)|/2$. We have the following lemma which is an immediate consequence of (10)

Lemma 5 ([13]) *If G is a graph with n vertices, then*

1. *the bisection width of G is at least $n\theta_2(G)/4(1 + o(1))$, and*
2. *$bip(X) \leq n\theta_n(G)/4$.*

4. Properties of quadrance graphs

The following theorem summarizes some of the properties of quadrance graphs.

Theorem 6 *Let q be any prime of the form $q = 12k + 7$ with $k \geq 0$ and $a \neq 0$. The quadrance graph $D_q(a)$ is a $d_q = (q + 1)$ -regular graph on $n_q = q^2$ vertices with the following properties.*

1. $D_q(a)$ is triangle-free and has diameter 3.
2. The toughness of $D_q(a)$ is at least $q^{1/2}/6 = n_q^{1/4}/6$.
3. The bisection width of $D_q(a)$ is at least $q^2(q - 2q^{1/2})/(4 + o(1))$.
4. The maximum number of edges in a spanning bipartite subgraph of $D_q(a)$ is at most $\text{bip}(D_q(a)) \leq q^2(q + 2q^{1/2})/4$.
5. The independence number of $D_q(a)$ is at most $2q^{3/2} = 2n_q^{3/4}$, and hence its chromatic number is at least $n_q^{1/4}/2$.

Proof From Lemma 1, $D_q(a)$ has no triangle as $f(a, a, a) = 3a^2/4$ is a non-square in \mathbb{F}_q (If $q = 12k + 7$ then 3 is a non-square in \mathbb{F}_q). Part 1 now follows from Theorem 2.

Part 2 follows directly from Lemma 3 and Theorem 5. Parts 3 and 4 are immediate from Lemma 3 and Lemma 5.

Part 5 follows easily from (10) as follows. Let S be a maximum independent set of $D_q(a)$, then $|\delta S| = |S|(q + 1)$. From (10), we have

$$\frac{q^2|S|(q + 1)}{|S|(q^2 - |S|)} \leq \theta_n(D_q(a)) \leq q + 1 + 2q^{1/2}.$$

This implies that the independence number $\alpha(D_q(a))$ is at most

$$\alpha(D_q(a)) = |S| \leq q^2 - \frac{q^2(q + 1)}{q + 1 + 2q^{1/2}} \leq 2q^{3/2} = 2n_q^{3/4}. \quad (11)$$

Hence, the chromatic number $\chi(D_q(a))$ is at least

$$\chi(D_q(a)) \geq \frac{|V(D_q(a))|}{\alpha(D_q(a))} \geq \frac{n_q}{2n_q^{3/4}} = n_q^{1/4}/2. \quad (12)$$

This concludes the proof of the theorem. \square

Theorem 6 shows that the quadrance graph $D_q(a)$, where q is a prime of the form $q = 12k + 7$ and $a \neq 0 \in \mathbb{F}_q$, is an explicit triangle-free graph on $n_q = q^2$ vertices whose chromatic number exceeds $0.5n_q^{1/4}$. Note that the lower bound was already observed in [19]. In addition, the quadrance graph $D_q(a)$ is an explicit construction showing that $R(3, k) \geq \Omega(k^{4/3})$. Moreover, $D_q(a)$ has $q^2(q + 1)/2$ edges so the graph $D_q(a)$ is also an explicit construction of a triangle-free graph G with e edges and

$$\text{bip}(G) \leq \frac{e}{2} + \frac{1}{2}e^{5/6}. \quad (13)$$

In general, the quadrance can be defined in any m -dimensional space \mathbb{F}_q^m for $m \geq 2$ as follows.

Definition 5 *The quadrance $Q(X, Y)$ between the points $X = (x_1, \dots, x_m)$ and $Y = (y_1, \dots, y_m)$ in \mathbb{F}_q^m is the number*

$$Q(X, Y) := \sum_{i=1}^m (x_i - y_i)^2.$$

For any fixed $a \neq 0 \in \mathbb{F}_q$, the quadrance graph $D_q^m(a)$ has the vertex set \mathbb{F}_q^m , and $X, Y \in \mathbb{F}_q^m$ are adjacent if and only if $Q(X, Y) = a$. Similarly to the above, we have the following theorem.

Theorem 7 *Let q be any odd prime power and $a \neq 0 \in \mathbb{F}_q$. The unit-quadrance graph $D_q^m(a)$ is a $d_{q,m}$ -regular graph on $n_{q,m} = q^m$ vertices with the following properties.*

1. ([17]) Let χ be the quadratic character. Equivalently, χ is 1 on squares, 0 at 0 and -1 otherwise. Then

$$d_{q,m} = \begin{cases} q^{m-1} + \chi((-1)^{(m-1)/2})q^{(m-1)/2} & \text{if } m \text{ is odd,} \\ q^{m-1} - \chi((-1)^{n/2})q^{(n-2)/2} & \text{otherwise.} \end{cases}$$

2. ([17]) Let $\lambda \neq \Delta(D_q^m)$ be any eigenvalue of the graph D_q^m . Then

$$|\lambda| \leq 2q^{(m-1)/2}.$$

3. The toughness of D_q is at least

$$q^{(m-1)/2}/(6 + o(1)) = n_{q,m}^{(m-1)/2m}/(6 + o(1)).$$

4. The bisection width of D_q is at least $q^m(d_{q,m} - 2q^{(m-1)/2})/(4 + o(1))$.

5. The maximum number of edges in a spanning bipartite subgraph of D_q is at most $\text{bip}(D_q) \leq q^m(d_{q,m} + 2q^{(m-1)/2})/4$.

6. The independence number of D_q is at most $(2 + o(1))q^{(m+1)/2} = (2 + o(1))n_{q,m}^{(m+1)/2m}$, and hence its chromatic number is at least $n_q^{(m-1)/2m}/(2 + o(1))$

The proof of this theorem is omitted since it is the same as the proof of Theorem 6. Note that $D_q^m(a)$ is triangle-free if and only if $m = 2$ and q is a prime of the form $q = 12k \pm 5$. Thus Theorem 7 is not more useful than Theorem 6 in attacking triangle-free graphs.

5. Properties of finite non-Euclidean graphs

The following theorem summarizes some of the properties of finite non-Euclidean graphs.

Theorem 8 Let q be any prime of the form $q = 12k + 5$ with $k \geq 1$. The finite non-Euclidean graph $V_q(3, 6)$ is a $d_q = (q + 1)$ -regular graph on $n_q = q^2 - q$ vertices with the following properties.

1. $V_q(3, 6)$ is triangle-free and has diameter 3.
2. The toughness of $V_q(3, 6)$ is at least $q^{1/2}/6 > n_q^{1/4}/6$.
3. The bisection width of $V_q(3, 6)$ is at least $(q^2 - q)(q - 2q^{1/2})/(4 + o(1))$.
4. The maximum number of edges in a spanning bipartite subgraph of $V_q(3, 6)$ is at most $\text{bip}(V_q(3, 6)) \leq (q^2 - q)(q + 2q^{1/2})/4$.
5. The independence number of $V_q(3, 6)$ is at most $(2 + o(1))n_q^{3/4}$, and hence its chromatic number is at least $n_q^{1/4}/(2 + o(1))$

Proof Part 1 follows from Theorems 3 and 4 combined with the fact that 3 is not a square in \mathbb{F}_q if $q = 12k + 5$.

Part 2 follows directly from Lemma 4 and Theorem 5.

Parts 3 and 4 are immediate from Lemma 4 and Lemma 5.

The proof of Part 5 is similar to the proof of Theorem 6, Part 5. □

Theorem 8 shows that the finite non-Euclidean graph $V_q(3, 6)$, where q is a prime of the form $q = 12k + 5$, is an explicit triangle-free graph on $n_q = q^2 - q$ vertices whose chromatic number exceeds $(0.5 + o(1))n_q^{1/4}$. In addition, the finite non-Euclidean graph $V_q(a)$ is an explicit construction showing that $R(3, k) \geq \Omega(k^{4/3})$.

6. Further remarks

It is known that the chromatic number of any graph with maximum degree d in which the number of edges in the induced subgraph on the set of all neighbors of any vertex does not exceed d^2/f is at most $O(d/\log f)$ (see [6]). Let q be a prime of the form $q = 12k \pm 5$. Then the induced subgraph on the set of all neighbors of any vertex is empty. So we can set $f = q^2$. For any q , the induced subgraph on the set of all neighbors of any vertex of $D_q(a)$ has at most $q + 1$ edges. This implies that we can set $f = d - 1$. This implies the following upper bound for the chromatic number of quadrance graphs $D_q(a)$

$$\chi(D_q(a)) \leq O(q/\log_2 q). \quad (14)$$

This method also gives us a similar bound for higher dimensional cases.

$$\chi(D_q^m(a)) \leq O(q^{m-1}/\log_2 q). \quad (15)$$

Note that the best known constructive upper bound for the chromatic number of general quadrance graphs is much weaker:

$$\chi(D_q^m) \leq q^{m-1}(1/2 + o(1)). \quad (16)$$

This bound is obtained by an explicit coloring in [25].

Finally, the bounds in Theorems 6 and 8 match the bounds obtained by code graphs in Theorem 3.1 in [4]. These latter graphs are Cayley graphs and their construction is based on some of the properties of certain dual BCH error-correcting codes. For a positive integer k , let $F_k = GF(2^k)$ denote the finite field with 2^k elements. The elements of F_k are represented by binary vectors of length k . If a and b are two such vectors, let (a, b) denote their concatenation. Let G_k be the graph whose vertices are all $n = 2^{2k}$ binary vectors of length $2k$, where two vectors u and v are adjacent if and only if there exists a non-zero $z \in F_k$ such that $u + v = (z, z^3) \bmod 2$ where z^3 is computed in the field F_k . Then G_k is a $d_k = 2^k - 1$ -regular graph on $n_k = 2^{2k}$ vertices. Moreover, G_k is triangle-free with independence number at most $2n^{3/4}$. Alon gives the better bound $R(m, 3) \geq \Omega(m^{3/2})$ in [3] by considering a graph whose vertex set is the set of all $n = 2^{3k}$ binary vectors of length $3k$ (instead of all binary vectors of length $2k$). Suppose that k is not divisible by 3. Let W_0 be the set of all nonzero elements $\alpha \in F_k$ such that the leftmost bit in the binary representation of α^7 is 0, and let W_1 be the set of all nonzero elements $\alpha \in F_k$ for which the leftmost bit of α^7 is 1. Then $|W_0| = 2^{k-1} - 1$ and $|W_1| = 2^{k-1}$. Let G_n be the graph whose vertices are all $n = 2^{3k}$ binary vectors of length $3k$, where two vectors u and v are adjacent if and only if there exist $w_0 \in W_0$ and $w_1 \in W_1$ such that $u + v = (w_0, w_0^3, w_0^5) + (w_1, w_1^3, w_1^5)$ where the powers are computed in the field F_k and the addition is addition modulo 2. Then G_n is a $d_n = 2^{k-1}(2^{k-1} - 1)$ -regular graph on $n = 2^{3k}$ vertices. Moreover, G_n is a triangle-free graph with independence number at most $(36 + o(1))n^{2/3}$. Note that, going to higher dimensional unit-quadrance graphs does not give us a better bound for the Ramsey number $R(3, k)$ as for any $m \geq 3$ the graph $D_q^m(a)$ is not triangle-free. The problem of finding better bounds for the chromatic number of the graphs $D_q(a)$ and $V_q(\sigma, a)$ touches on an important question in graph theory: what is the greatest possible chromatic number for a triangle-free regular graph of order n ? A possible approach is to consider the existence of sum-free varieties in high dimensional vector spaces over finite fields. We see that the varieties of degree two only give us triangle-free graphs on vector spaces of dimension two. We hope to address this problem for varieties of higher degrees in higher dimensions in a subsequent paper.

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